e^x and the differential of 10^x

Introduction

The Feynman lectures in Physics contains a numerical example for calculating the power of 10^x, (see 22-5 and table 22-2). With modern computing methods it is easy to miss how important these techniques were historically. As an exercise to explain powers and exponential, Feynman used a numerical approximation. This approximation was historically important for the calculation of logarithmic tables that were used for some three hundred years,

In the example it is shown numerically that the differential of 10^x tends to a constant value for small values of x. This simplifies finding small powers of 10^x from an operation of taking a power to a simpler operation involving only multiplication.

These notes were made to clarify why it tends to a limit and how this relates to the exponential.

Expansion as a finite difference for y^x

Consider the differential $\frac{d10^x}{\mathrm{dx}}$ expanded as the finite difference $\frac{10^{(x_2 - x_1)} - 10^{x_1}}{\Delta x}$

Numerical values range from zero to x_2, so $x_1 = 0$ and $\frac{10^{\Delta x} - 1}{\Delta x}$

10^x is a smooth function and can be expanded as a polynomial series,

$$10^x = 1 + \mathrm{ax} + \mathrm{bx}^2 + c^3 \ldots$$

For small x, the higher powers will vanish and the difference reduces to $1 + a\Delta x$. Substituting this into the differential equation shows that, in the limit of small x, it tends to a constant value,

$$\lim_{x \to 0} \frac{d10^x}{\mathrm{dx}} = a_{\mathrm{const}}$$

This result is general to any base, y, and in the limit,

$$\lim_{x \to 0} \frac{\mathrm{dy}^x}{\mathrm{dx}} = b_{\mathrm{const}}$$

Summary

This answers why the numerical example tends to a constant value. It is the value of a_{const} that Feynman finds by numerical means. If a_{const}, or in general b_{const}, is known, then for small x it is easy to find the power,

$$\lim_{x \to 10} 10^x = 1 + a_{\mathrm{const}} x \text{ , or in general, } y^x = 1 + b_{\mathrm{const}} x$$

y^x and the exponential

In the previous section it was shown that, in the limit, y^x will tend to a constant value. It is possible to take this further and find this limiting value and how it relates to other bases. Consider what happens if y^x is re-expressed in terms of some new base z^x,

$$y^x = z^{\log_z^{y^x}} = z^{\mathrm{x} \log_z (y)} = \lim_{x \to 0} 1 + \log_z (y) x$$

From the previous section it was found that $\lim_{x\to 0} y^x = 1 + b_{\text{const}}x$. This gives a relation between bases in terms of the limiting value b_{const},

$$b_{\text{const}} = \log_z(y)$$

The most natural base to work with would be one where b_{const} =1. This is called the exponential base,

$$\lim_{x\to 0} e^x = 1 + x$$

Knowing this, other bases can be scaled to it,

$$10^x = e^{\ln 10^x} = e^{\text{xln10}} = \lim_{x\to 0} 1 + \ln(10)x$$

It is this value of ln(10), 2.3025, that Feynman's series is tending to in table 22-2.